RANDOM PRODUCTS OF CONTRACTIONS IN BANACH SPACES

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ABSTRACT. We show that the random product of a finite number of (W) contractions converges weakly in all smooth reflexive Banach spaces. If one of the contractions is compact, then the convergence is uniform.

Let $(X, |\cdot|)$ be a (real) Banach space and $T\colon X \to X$ a linear operator. Recall that T is called a contraction if $|Tv| \leq |v|$ for all vectors v in X. We say that a contraction T satisfies condition (W) if whenever $\{v_n\}$ is bounded and $|v_n| - |Tv_n| \to 0$, it follows that the weak $\lim_{n\to\infty}(v_n - Tv_n) = 0$. The algebraic semigroup $\mathbf{S}$ generated by a (possibly infinite) set of contractions will be said to satisfy condition (W) if for any bounded sequence of vectors $\{v_n\} \subset X$ and a sequence of words $\{W_n\}$ from $\mathbf{S}$ such that $|v_n| - |W_n v_n| \to 0$, the weak $\lim_{n\to\infty}(v_n - W_n v_n) = 0$.

Now let $\{T_1, T_2, \ldots\}$ be a sequence of contractions that satisfy condition (W), and let r be a self-mapping of the set of natural numbers. A random product of these contractions is the sequence $\{S_n : n = 1, 2, \ldots\}$ defined by

$$S_n = T_{r(n)} T_{r(n-1)} \cdots T_{r(1)}.$$

Our purpose in this paper is to study the convergence properties of such products. Such a study is not only of intrinsic interest, but is also motivated by applications to the numerical solution of partial differential equations [2, 14] and linear inequalities [19], approximation theory [16, 9] and computer tomography [18, 11].

We begin by noting several properties of contractions which satisfy condition (W). We shall call such contractions (W) contractions for short. We then consider the algebraic semigroup generated by a finite number of (W) contractions and establish a weak convergence theorem for random products of these contractions in all smooth reflexive Banach spaces (Theorem 1). This seems to be the first such result outside Hilbert space. It includes the Hilbert space theorem due to Amemiya and Ando [1]. We continue with a weak convergence result for semigroups generated by an infinite sequence of contractions (Theorem 2). Finally, we obtain a uniform convergence theorem for random

products of compact contractions (Theorem 3). We first observe (cf. [1]) the following fact.

Lemma 1. *The class of* (W) *contractions is closed under composition.*

Proof. Let T_1 and T_2 satisfy condition (W) and suppose that $\{v_n\}$ is bounded and $|v_n| - |T_2 T_1 v_n| \to 0$. Since $|T_2 T_1 v_n| \leq |T_1 v_n| \leq |v_n|$, we also have $|v_n| - |T_1 v_n| \to 0$ and $|T_1 v_n| - |T_2 T_1 v_n| \to 0$. Hence

$$\text{the weak } \lim_{n \to \infty} (v_n - T_1 v_n) = \text{the weak } \lim_{n \to \infty} (T_1 v_n - T_2 T_1 v_n) = 0.$$

The result follows.

We denote the range of an operator T by $R(T)$ and its fixed point set by $F(T)$. In the sequel we shall repeatedly use the following simple observation: If $\{v_n\}$ is a bounded sequence of vectors in a reflexive Banach space X which does not converge weakly to 0, then there is a subsequence $\{w_k\}$ of $\{v_n\}$ such that $w_k = u + q_k$, where $u \neq 0$ and $\{q_k\}$ converges weakly to 0.

Proposition 1. *If* X *is a reflexive Banach space and the contraction* $T: X \to X$ *satisfies* (W), *then for each* x *in* X, $\{T^n x\}$ *converges weakly to a fixed point of* T.

Proof. Recall that the mean ergodic theorem provides us with a contractive projection $Q: X \to F(T)$ such that $TQ = QT = Q$. the sequence Therefore we may restrict our attention to x in $R(I - Q)$, a closed subspace of X which is invariant under T. If $\{T^n x\}$ does not converge weakly to 0, then there is a subsequence of $\{T^n x\}$, which we denote by x_k, such that $x_k = u + q_k$, where $u \neq 0$ and $\{q_k\} \to 0$ weakly. Since T is a (W) contraction, we have $\lim_{n \to \infty} |T^n x| = \lim_{n \to \infty} |T^{n+1} x|$ and the weak $\lim_{n \to \infty} (T^n x - T^{n+1} x) = 0$. Hence $0 = \text{the weak } \lim_{n \to \infty} (x_k - Tx_k) = \text{the weak } \lim_{n \to \infty} (u - Tu + q_k - Tq_k) = u - Tu$. Thus u belongs to $F(T) \cap R(I - Q) = \{0\}$. The contradiction we have reached shows that $\{T^n x\}$ does converge weakly to 0 and the proof is complete.

Alternatively, we note that for any x in X and $k \geq 0$,

$$\left| \left(\sum_{i=1}^{n} T^{i+k} x \right) / n - Qx \right| = \left| \left(\sum_{i=1}^{n} T^{i+k} x - T^k Qx \right) / n \right|$$

$$\leq \left| \left(\sum_{i=1}^{n} T^{i} x \right) / n - Qx \right|.$$

Therefore the strong $\lim_{n \to \infty} (\sum_{i=1}^{n} T^i x)/n = Qx$, uniformly in $k \geq 0$. But we also know that the weak $\lim_{n \to \infty} (T^n x - T^{n+1} x) = 0$ and this is a Tauberian condition for almost convergence. Hence $\{T^n x\}$ converges weakly to Qx.

In order to proceed we need the following simple lemma (cf. [1]).

Lemma 2. *If* $\{T_j : 1 \leq j \leq N\}$ *are* (W) *contractions, then*

$$\bigcap \{F(T_j) : 1 \leq j \leq N\} = F(T_N T_{N-1} \cdots T_1).$$

Proof. It is clear that $\bigcap\{F(T_j) : 1 \le j \le N\} \subset F(T_N T_{N-1} \cdots T_1)$. Conversely, suppose $x \in F(T_N T_{N-1} \cdots T_1)$ and let $y = T_{N-1} \cdots T_1 x$. Then $|x| = |T_N y| \le |y| \le |x|$. Since T_N satisfies condition (W), $y = T_N y$. Hence $T_{N-1} \cdots T_1 x = x$, $T_N x = x$, and the lemma follows.

Our next result is obtained by combining Proposition 1 with Lemmas 1 and 2.

Proposition 2. *Let* $\{T_j : 1 \le j \le N\}$ *be* N (W) *contractions on a reflexive Banach space* X. *Then the weak* $\lim_{n \to \infty} (T_N T_{N-1} \cdots T_1)^n x = Qx$ *exists for all* x *in* X *and defines a contractive projection* Q *of* X *onto* $\bigcap\{F(T_j) : 1 \le j \le N\}$.

It is known [17, Lemma 3.2; 15, p. 162] that if E is a closed subspace of a smooth Banach space X, then there is at most one nonexpansive retraction of X onto E. In the setting of Proposition 2 this means that if X is also smooth, then the contractive projection Q is the only nonexpansive retraction of X onto $\bigcap\{F(T_j) : 1 \le j \le N\}$. Since QT_j is also such a retraction, it follows that $QT_j = Q = T_j Q$ for each $1 \le j \le N$, and that X is the direct sum of the closed subspaces $R(Q)$ and $R(I - Q)$, both of which are invariant under each T_j, $1 \le j \le N$.

We say that a contraction T satisfies condition (W') if $|v| = |Tv|$ implies that $v = Tv$. (See [10].) Clearly a (W) contraction also satisfies condition (W'). It turns out that in some spaces these two conditions are equivalent.

Let $\varphi : R^+ \to R^+$ be continuous and strictly increasing, with $\varphi(0) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$. Recall that the duality map J_φ from a Banach space X into the family of nonempty (by the Hahn-Banach Theorem) subsets of its dual X^*, corresponding to the gauge function φ, is defined by

$$J_\varphi(x) = \{x^* \in X^* : (x, x^*) = |x||x^*| \text{ and } |x^*| = \varphi(|x|)\}.$$

This duality map is single-valued if X is smooth. It is said to be weakly sequentially continuous if X is smooth and whenever the sequence $\{x_n\}$ converges weakly to x, $\{J_\varphi(x_n)\}$ converges weak-star to $J_\varphi(x)$. This is the case, for example, for all l_p spaces, $1 < p < \infty$.

Proposition 3. *If the reflexive Banach space has a weakly sequentially continuous duality map, then the conditions* (W) *and* (W') *are equivalent.*

Proof. Let $T: X \to X$ be a contraction that satisfies condition (W'), and assume that $\{v_n\} \subset X$ is a bounded sequence such that $|v_n| - |Tv_n| \to 0$. If $\{v_n - Tv_n\}$ does not converge weakly to 0, then $\{v_n\}$ has a subsequence, which we continue to denote by $\{v_n\}$, that converges weakly to $u \notin F(T)$. We may also assume that $\lim_{n \to \infty} |v_n|$ exists. We write $v_n = u + q_n$, where $q_n \to 0$ weakly. Now let $J_\varphi : X \to X^*$ be a weakly sequentially continuous duality map with a gauge function $\varphi : R^+ \to R^+$, which is continuous and strictly increasing, with $\varphi(0) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$, and set $\Phi(t) = \int_0^t \varphi(r) \, dr$.

Since $\Phi(|x+y|) = \Phi(|x|) + \int_0^1 (y, J_\varphi(x+ty))\,dt$ for all x and y in X, we have

$$\Phi(|u+q_n|) = \Phi(|q_n|) + \int_0^1 (u, J_\varphi(q_n+tu))\,dt$$

and

$$\Phi(|Tu+Tq_n|) = \Phi(|Tq_n|) + \int_0^1 (Tu, J_\varphi(Tq_n+tTu))\,dt.$$

Therefore,

$$\int_0^1 (u, J_\varphi(tu))\,dt - \int_0^1 (Tu, J_\varphi(tTu))\,dt$$

$$= \lim_{n\to\infty} \left[\int_0^1 (u, J_\varphi(q_n+tu))\,dt - \int_0^1 (Tu, J_\varphi(Tq_n+tTu))\,dt\right]$$

$$\leq \lim_{n\to\infty} \left[\int_0^1 (u, J_\varphi(q_n+tu))\,dt - \int_0^1 (Tu, J_\varphi(Tq_n+tTu))\,dt\right.$$

$$\left. + \Phi(|q_n|) - \Phi(|Tq_n|)\right]$$

$$= \lim_{n\to\infty} [\Phi(|u+q_n|) - \Phi(|Tu+Tq_n|)] = 0.$$

In other words,

$$|u|\int_0^1 \varphi(|tu|)\,dt \leq |Tu|\int_0^1 \varphi(t|Tu|)\,dt,$$

or $\Phi(|u|) \leq \Phi(|Tu|)$. Hence $|Tu| = |u|$ and $Tu = u$ by condition (W'). This contradiction completes the proof.

Combining Propositions 1 and 3 we obtain the following improvement of [5, Corollary 8.1].

Proposition 4. *If X is a reflexive Banach space with a weakly sequentially continuous duality map and the contraction $T: X \to X$ satisfies condition (W'), then for each x in X, $\{T^n x\}$ converges weakly to a fixed point of T.*

Conditions (W) and (W') are not equivalent in all reflexive Banach spaces. To see this, consider the space l^2 renormed by $|x| = \max\{|x|_2, 2^{1/2}|x|_\infty\}$ and the contraction T defined by $(Tx)_n = a_n x_n$, where $0 \leq a_n < 1$ and $\{a_n\}$ increases to 1. Set $v_n = e_1 + e_n$ where e_n is the nth unit vector. Then $(|v_n| - |Tv_n|) \to 0$, but the weak $\lim_{n\to\infty}(v_n - Tv_n) \neq 0$. On the other hand, there are Banach spaces (such as the l_p sum of a sequence of finite-dimensional nonsmooth spaces) which do not have a weakly sequentially continuous duality map, but in which conditions (W) and (W') are equivalent. To see this, note that what is required in the proof of Proposition 3 is the following property of a Banach space X (cf. [13, p. 137]):

If $|q_n| \to s$ and $q_n \to 0$ weakly, then $\lim_{n\to\infty} |u+q_n| = \delta(|u|, s)$ for all u in X, where $\delta: R^+ \times R^+ \to R^+$ is strictly increasing in the first variable

and increasing in the second. (In the case of Proposition 3, $\delta(r, s) = \Phi^{-1}(\Phi(r) + \Phi(s))$.)

We remark in passing that Propositions 1 and 2 have nonlinear analogs. Let C be a closed convex subset of a Banach space X. Recall that an operator $T: C \to C$ is said to be nonexpansive if $|Tx - Ty| \leq |x - y|$ for all x and y in C. We say that a nonexpansive mapping T satisfies condition (W) if whenever $\{x_n - y_n\}$ is bounded and $(|x_n - y_n| - |Tx_n - Ty_n|) \to 0$ it follows that the weak $\lim_{n \to \infty}((x_n - y_n) - (Tx_n - Ty_n)) = 0$.

Proposition 5. *Let $\{T_j : 1 \leq j \leq N\}$ be N nonexpansive self-mappings of a closed convex bounded subset C of a Banach space X. If each T_j satisfies condition (W) and X is uniformly convex with a Fréchet differentiable norm, then the weak $\lim_{n \to \infty}(T_N T_{N-1} \cdots T_1)^n x = Qx$ exists for all x in C and defines a nonexpansive retraction of C onto $\bigcap\{F(T_j) : 1 \leq j \leq N\}$.*

Proof. Let y be a fixed point of $T = T_N T_{N-1} \cdots T_1$. Then

$$(|T^n x - y| - |T^{n+1}x - y|) \to 0.$$

Since T also satisfies condition (W), the weak $\lim_{n \to \infty}(T^n x - T^{n+1}x) = 0$. Since this is a Tauberian condition for weak almost convergence, the result is now seen to follow from the nonlinear mean ergodic theorem.

Since all strongly nonexpansive mappings in the sense of [6] satisfy condition (W), we see that in a uniformly convex space all averaged and firmly nonexpansive mappings (including all linear contractive projections and all resolvents of accretive operators) satisfy condition (W). We also note that all weakly compact convex subsets of Banach spaces have the fixed point property for nonexpansive mapping which satisfy condition (W).

We now turn out attention to semigroups.

Proposition 6. *Let $\{T_j : 1 \leq j \leq N\}$ be N (W) contractions on a smooth reflexive Banach space. Then the algebraic semigroup $\mathbf{S} = \mathbf{S}(T_1, T_2, \ldots, T_N)$ generated by them also satisfies condition (W).*

Proof. Let Q_k be the unique contractive projection of the Banach space X onto $\bigcap\{F(T_j) : 1 \leq j \leq k\}$, $1 \leq k \leq N$, which is provided by Proposition 2 and the remarks following it. In the sequel we shall decompose the v_n's which appear in the definition of condition (W) as $v_n^{(1)} + v_n^{(2)}$, where $v_n^{(1)} \in Q_k X$ and $v_n^{(2)} \in (I - Q_k)X$, for an appropriate k.

We proceed by induction, considering the case $N = 1$ first. For some bounded v_n and some self-mapping $k(n)$ of the set of natural numbers for which

(1) $$|v_n| - |T^{k(n)}v_n| \to 0,$$

we wish to show that $v_n - T^{k(n)}v_n \to 0$ weakly. Since $Tv_n^{(1)} = v_n^{(1)}$, it suffices to show that

(2) $$(I - T^{k(n)})v_n^{(2)} \to 0 \text{ weakly}.$$

We can drop to a subsequence and assume $k(n) > 1$. In light of (1), we have $|v_n| - |Tv_n| \to 0$, which implies that

$$v_n^{(2)} - Tv_n^{(2)} \to 0 \text{ weakly},\tag{3}$$

as T has (W). If $v_n^{(2)}$ does not converge weakly to zero, we may assume for a subsequence that $v_n^{(2)} = u + q_n$, where $u \neq 0$ and $q_n \to 0$ weakly. Hence (3) becomes

$$u + q_n - Tu - Tq_n \to 0 \text{ weakly}.\tag{4}$$

The closed subspace $(I-Q)X$ (where $Q = Q_1$) is invariant for T, and includes its weak limits. Hence, $u \in (I - Q)X$. But (4) implies $Tu = u$. We conclude that $u = 0$, and this contradiction implies that

$$v_n^{(2)} \to 0 \text{ weakly}.\tag{5}$$

Using (1) again, we see that $|T^{k(n)-1}v_n| - |T^{k(n)}v_n| \to 0$, which implies that $T^{k(n)-1}v_n - T^{k(n)}v_n \to 0$ weakly. Hence

$$T^{k(n)-1}v_n^{(2)} - T^{k(n)}v_n^{(2)} \to 0 \text{ weakly}.\tag{6}$$

If $T^{k(n)-1}v_n^{(2)}$ does not converge weakly to zero, then for a subsequence, $T^{k(n)-1}v_n^{(2)} = u + q_n$, where $u \neq 0$ and $q_n \to 0$ weakly. Then (6) becomes

$$u + q_n - Tu - Tq_n \to 0 \text{ weakly}.\tag{7}$$

As before, (7) implies that $Tu = u$, which implies that $u = 0$, and this contradiction implies that $T^{k(n)}v_n^{(2)} \to 0$ weakly. This fact and (5) imply that $v_n^{(2)} - T^{k(n)}v_n^{(2)} \to 0$ weakly. We conclude that $v_n - T^{k(n)}v_n \to 0$ weakly.

Proceeding inductively, suppose the Proposition is valid for words from the first $k - 1 < N$ letters. Say there exist bounded vectors v_n and words $W_n \in S(T_1, \dots, T_k)$ such that

$$|v_n| - |W_n v_n| \to 0.\tag{8}$$

We wish to show that $v_n^{(2)} - W_n v_n^{(2)} \to 0$ weakly. (Here $v_n^{(2)} \in (I - Q_k)X$.) By induction and a possible drop to a subsequence, we may further assume that the W_n's are complete (each contains all k letters). Suppose $v_n^{(2)}$ does not converge weakly to zero. We may drop to a subsequence and assume

$$v_n^{(2)} = u + q_n, \text{ where } u \neq 0 \text{ and } q_n \to 0 \text{ weakly}.\tag{9}$$

An easy combinatorial argument (and a possible reindexing of the T_j's, $j = 1, \dots, k$) allows us to assume that (for some subsequence) there exist words F_n and T_k such that $W_n = (\cdots)T_k F_n$, where the $F_n \in \mathbf{S}(T_1, \dots, T_{k-1})$ and are complete (in $T_1, \dots, T_{k-1}$).

Our immediate goal is to show that u is fixed by the F_n. If u is not fixed, note the first letters T_n in F_n for which u is *not* a fixed point. Denote by A_n

the interceding words (which fix u). (In the event the word A_n is null, replace it with the identity map I.) Since we are dealing with just $k - 1$ distinct maps, we may assume by dropping to a subsequence that $T_k F_n = (\cdots) T A_n$, for some *fixed* map T. Note that (8) implies that $|v_n| - |A_n v_n| \to 0$. Hence by induction and (9) we have

$$(10) \qquad u + q_n - A_n u - A_n q_n = q_n - A_n q_n \to 0 \text{ weakly},$$

which implies that

$$(11) \qquad A_n q_n \to 0 \text{ weakly}.$$

By (9) and the fact that T has (W) we have

$$(12) \qquad u + q_n - T A_n u - T A_n q_n = u - Tu + q_n - T A_n q_n \to 0 \text{ weakly}.$$

Hence (11) implies that $u = Tu$, a contradiction. Thus $F_n u = u$.

By (8), $|F_n v_n| - |T_k F_n v_n| \to 0$. As T_k has (W), we must have that $F_n v_n - T_k F_n v_n \to 0$ weakly. This implies that $F_n v_n^{(2)} - T_k F_n v_n^{(2)} \to 0$ weakly. Using (9), we have $u + F_n q_n - T_k u - T_k F_n q_n \to 0$ weakly. Since $F_n = A_n$, we have $F_n q_n \to 0$ weakly and $T_k F_n q_n \to 0$ weakly by (11). Hence u is fixed by T_k, as well as the other $k - 1$ maps. This means that $u = 0$, as $u \in (I - Q_k) X$. We conclude that

$$(13) \qquad v_n^{(2)} \to 0 \text{ weakly}.$$

We will be done if we show that $W_n v_n^{(2)} \to 0$ weakly. Suppose that $W_n v_n^{(2)}$ does not converge weakly to zero. Then by dropping to a subsequence we may assume that $W_n v_n^{(2)} \to z \neq 0$, weakly. In a similar manner as before, a combinatorial argument (and a possible reindexing of the T_j's, $j = 1, \ldots, k$) allows us to assume that (for some subsequence) there exist words $F_n T_k$ such that $W_n = F_n T_k (\cdots)$, where the $F_n \in \mathbf{S}(T_1, \ldots, T_{k-1})$ and are complete (in $T_1, \ldots, T_{k-1}$). Suppose $(\cdots) v_n^{(2)}$ does not converge weakly to zero. We drop to a subsequence for which

$$(14) \qquad (\cdots) v_n^{(2)} = u + q_n, \text{ where } u \neq 0 \text{ and } q_n \to 0 \text{ weakly}.$$

By (8), $|(\cdots) v_n| - |T_k (\cdots) v_n| \to 0$. Since T_k has (W), we have that $(\cdots) v_n - T_k (\cdots) v_n \to 0$ weakly. Hence $(\cdots) v_n^{(2)} - T_k (\cdots) v_n^{(2)} \to 0$ weakly. In light of (14), we have $u + q_n - T_k u - T_k q_n \to 0$ weakly. So u is fixed by T_k. Our goal, as before, is to show that u is fixed by all the other $k - 1$ maps. Let A_n and T be defined as before in the proof. Now $|T_k (\cdots) v_n| - |A_n T_k (\cdots) v_n| \to 0$, so by induction we have $T_k (\cdots) v_n - A_n T_k (\cdots) v_n \to 0$ weakly, and hence $T_k (\cdots) v_n^{(2)} - A_n T_k (\cdots) v_n^{(2)} \to 0$ weakly. Substituting into (14), we have $u + T_k q_n - A_n u - A_n T_k q_n \to 0$ weakly. This implies that

$$(15) \qquad A_n T_k q_n \to 0 \text{ weakly}.$$

Now T has (W), so that, using (8), we have $u + A_n T_k q_n - Tu - T A_n T_k q_n \to 0$ weakly. By (15) we see that u is necessarily fixed by T. Recalling that

$u \in (I - Q_k)X$, we must have that $u = 0$, a contradiction. Hence $(\cdots)v_n^{(2)} \to 0$ weakly. But (15) and $u = 0$ imply that $A_n T_k(\cdots)v_n^{(2)} \to 0$ weakly. Clearly $T_k(\cdots)v_n^{(2)} \to 0$ weakly, because $u = 0$ in (14). Hence $W_n v_n^{(2)} \to 0$ weakly, a contradiction, as $W_n v_n^{(2)} \to z \neq 0$. Hence $W_n v_n^{(2)} \to 0$ weakly, completing the proof.

Remark. Inspection of the proof shows that if $|v_n| - |W_n v_n| \to 0$ and the W_n are complete, them in fact *both* $v_n^{(2)}$ and $W_n v_n^{(2)}$ converge weakly to zero.

Theorem 1. *Let* $\{T_j : 1 \leq j \leq N\}$ *be* N (W) *contractions on a smooth reflexive Banach space* X. *Let* r *be a mapping of the set of natural numbers onto* $\{1, 2, \ldots, N\}$ *which assumes each value infinitely often and let* $S_n = T_{r(n)} T_{r(n-1)} \cdots T_{r(1)}$. *Then the weak* $\lim_{n\to\infty} S_n x = Qx$ *exists for each* x *in* X *and* Q *is the unique contractive projection of* X *onto* $\bigcap\{F(T_j) : 1 \leq j \leq N\}$.

Proof. It suffices to show that $S_n x \to 0$ weakly for $x \in (I - Q)X$. If $S_n x$ does not converge weakly to zero, then by dropping to a subsequence (which we continue to denote by S_n), we may assume $S_n x = u + q_n$, where $u \neq 0$ and $q_n \to 0$ weakly. Since $u \in (I - Q)X$, and each map occurs infinitely often, we may note the first letters T_n in the original sequence (succeeding the last letter in S_n) for which u is *not* a fixed point. Denote by A_n the interceding words (which fix u). (In the event the word A_n is null, replace it with the identity map I.) Since we are dealing with just N distinct maps, we may assume by dropping to a subsequence that $T_n A_n S_n = T A_n S_n$, for some *fixed* map $T \in \{T_1, \ldots, T_N\}$. Since we have $|S_n x| - |A_n S_n x| \to 0$, Proposition 6 enables us to conclude that $(I - A_n)S_n x = q_n - A_n q_n \to 0$ weakly. Hence $A_n q_n \to 0$ weakly. We also have $|A_n S_n x| - |T A_n S_n x| \to 0$. Since T has (W), we have that

$$(I - T)A_n S_n x = u + A_n q_n - Tu - T A_n q_n \to 0 \text{ weakly.}$$

Since $T A_n q_n \to 0$ weakly, we see that $u = Tu$. This contradiction completes the proof.

Alternatively, for any n, let C_n be a complete word immediately following S_n. These exist as each map occurs infinitely often. If $x \in (I - Q)X$, then $S_n x \in (I - Q)X$, by the remarks following Proposition 2. $\mathbf{S}(T_1, \ldots, T_N)$ has (W), by Proposition 6. Hence $|S_n x| - |C_n S_n x| \to 0$ implies $S_n x \to 0$ weakly, as the C_n are complete. (Here we have substituted $S_n x$ for $v_n^{(2)}$ and applied the remark at the end of the proof of Proposition 6.)

Remark. Using Theorem 1.11 on p. 80 of [12], we may replace smoothness by the assumption that the adjoints $\{T_j^* : 1 \leq j \leq N\}$ are also (W) contractions.

We now present a weak convergence theorem for random products of an infinite sequence of contractions. For other results on such products see [2, 4, and 8]. One may note that the random product of an infinite sequence of (W) contractions need not converge weakly. See [4, p. 330; 7].

Theorem 2. *Let $\{T_1, T_2, \ldots\}$ be contractions on a reflexive Banach space X such that the algebraic semigroup $\mathbf{S} = \mathbf{S}(T_1, T_2, \ldots)$ has (W). Let r be a mapping of the set of natural numbers into itself with the property that each range value which is assumed, is assumed infinitely often, and let $S_n = T_{r(n)} T_{r(n-1)} \cdots T_{r(1)}$. Then the weak $\lim_{n \to \infty} S_n x = P x$ exists for each x in X and P is a contractive projection of X onto $\bigcap_{n=1}^{\infty} \{F(T_{r(n)})\}$.*

Proof. Suppose there exist subsequences $\{n\}$ and $\{n'\}$ such that $S_n x \to u$ weakly and $S_{n'} x \to u'$ weakly. Select a subsequence $\{n''\}$ of $\{n\}$ such that there exists an element of $\{n'\}$ between any two consecutive elements of $\{n''\}$. For this new subsequence $S_{n''}$, let W_n denote a (possibly null) word separating, in the original sequence, $S_{n''}$ from the very next occurrence of an element of $\{n'\}$. Hence, for an appropriate subsequence of $\{n'\}$, we may assume that $S_{n'} x = W_n S_{n''} x$. Clearly $|S_{n''} x| - |W_n S_{n''} x| \to 0$. Now (W) for $\mathbf{S}$ implies that $S_{n''} x - W_n S_{n''} x \to 0$ weakly. As $\{n''\}$ is a subsequence of $\{n\}$, we have that $S_{n''} x \to u$ weakly. Thus $u = u'$. We conclude that the original sequence converges to u weakly. If $u \notin \bigcap_{n=1}^{\infty} \{F(T_{r(n)})\}$, and each map occurs infinitely often, there exists a fixed map T such that $Tu \neq u$, and a subsequence $S_n x$ with the property that the very next map in the original sequence is T. We have $|S_n x| - |T S_n x| \to 0$. This implies that $Tu = u$ as T has (W). This contradiction completes the proof.

Remark. If $\{T_1, T_2, \ldots\}$ is a sequence of (W) contractions on a smooth reflexive Banach space X, then there still exists a unique contractive projection onto $\bigcap \{F(T_j) : 1 \le j < \infty\}$. This is because the intersection is a nonexpansive retract by [3, Lemma 4] and the retraction onto it is unique (and, in fact, linear) by [17]. Alternatively, we may show directly that if X is reflexive and $T_j : X \to X$, $1 \le j < \infty$ are (W') contractions, then there is a (W') contractive projection onto $\bigcap \{F(T_j) : 1 \le j < \infty\}$. These projections coincide with the projection P obtained in Theorem 2 if X is smooth.

In the setting of Theorem 1, condition (W) is not strong enough to guarantee strong convergence (see [8]). We do, however, obtain uniform convergence if one of the T_j's is compact. First we need two lemmas.

Recall that a Banach space is said to have the Kadec-Klee property (KKP for short) if whenever $x_n \to x$ weakly with $|x_n| \to |x|$, it follows that $x_n \to x$ strongly. We say that a contraction satisfies condition (S) if whenever the sequence $\{v_n\}$ is bounded and $|v_n| - |Tv_n| \to 0$, it follows that the strong $\lim_{n \to \infty} (I - T) v_n = 0$.

Lemma 3. *For a compact contraction T on a reflexive Banach space X, the conditions (W') and (W) are equivalent. In the event X also has the KKP, they are also equivalent to condition (S).*

Proof. We always have $(S) \Rightarrow (W) \Rightarrow (W')$. So it suffices to show that $(W') \Rightarrow (W)$ $(\Rightarrow (S)$ if X has the KKP).

Let $\{v_n\} \subset X$ be a bounded sequence such that $|v_n| - |Tv_n| \to 0$. If $\{v_n - Tv_n\}$ fails to converge weakly to 0, then $\{v_n\}$ has a subsequence, which we continue to denote by $\{v_n\}$, which converges weakly to $u \notin F(T)$. We may also assume that $s = \lim_{n \to \infty} |v_n|$ exists. Since T is compact, $\{Tv_n\}$ converges strongly to Tu and

$$|Tu| = \lim_{n \to \infty} |Tv_n| = \lim_{n \to \infty} |v_n| = s.$$

But we also have

$$s = |Tu| \leq |u| \leq \lim_{n \to \infty} \inf |v_n| = s.$$

Hence $|u| = |Tu|$ and $u = Tu$ by condition (W'). This contradiction proves that $(W') \Rightarrow (W)$.

If X also has the KKP, then $\{v_n\}$ converges strongly to u because $|u| = \lim_{n \to \infty} |v_n|$. Thus the strong $\lim_{n \to \infty}(I - T)v_n = 0$ and the result follows.

Our next lemma generalizes the following observation.

Let X be a reflexive Banach space and let $\{T_1, \dots, T_N\}$ be N contractions on X, each satisfying (W'), which have no common nonzero fixed points. Assume at least one of the T_j's is compact and let $T = T_1 \cdots T_N$. Then $\|T\| < 1$.

Indeed, if we let $LC(X)$ denote all compact operators on X, then since $LC(X)$ is an ideal in $L(X)$, $T \in LC(X)$. An argument similar to the proof of Lemma 1 shows that T satisfies (W'). Since the norm is a continuous function and the image of the unit ball is compact, there exists an x with $|x| = 1$ such that $\|T\|$ $(= \sup_{|y| \leq 1} |Ty|) = |Tx|$. In any event, $\|T\| \leq 1$. If $\|T\| = 1$, then $|Tx| = |x|$. Thus, by (W'), $Tx = x$. By Lemma 2, $x \in \bigcap_{i=1}^{N} F(T_i)$. Since by assumption the T_j's have no common nonzero fixed points, this rules out $\|T\| = 1$, proving $\|T\| < 1$.

Lemma 4. *Let $\{T_1, \dots, T_N\}$ be N contractions on a smooth reflexive Banach space satisfying condition (W'). Assume T_1 is compact. Let $\{W_n\}$ be a sequence of complete words in the T_i. Let Q be the projection onto $\bigcap \{F(T_j) : 1 \leq j \leq N\}$. Then, on $(I - Q)X$, $\sup_n \|W_n\| < 1$.*

Proof. If $\sup_n \|W_n\| = 1$ on $(I - Q)X$, there exist x_n of norm one such that $\lim_n |W_n x_n| = 1$. Hence $|x_n| - |W_n x_n| \to 0$. Since $x_n \in (I - Q)X$, and the $\{T_j : 1 \leq j \leq N\}$ have (W) by Lemma 3, we may apply the remark following Proposition 6, and conclude that

$$(16) \qquad\qquad\qquad W_n x_n \to 0 \text{ weakly.}$$

We factor W_n as $W_n = U_n T_1 V_n$.

As T_1 is compact, we can assume $T_1 V_n x_n \to u \neq 0$ by dropping to a subsequence. Some first letter T in U_n (following T_1) moving u must occur infinitely often, as each map occurs infinitely often. If not, we would have $W_n x_n \to u$ strongly, contradicting (16). Letting A_n denote the separating

words, we have that

$$TA_nT_1V_nx_n \to Tu \text{ weakly},$$

and

$$A_nT_1V_nx_n \to u \text{ weakly}.$$

Clearly $|A_nT_1V_nx_n| - |TA_nT_1V_nx_n| \to 0$. By (W) for T, we must have that $Tu = u$. This contradicts the construction of T, and finishes the proof.

Theorem 3. *Let* $\{T_1, \dots, T_N\}$ *be contractions on a smooth reflexive Banach space* X, *each satisfying condition* (W'). *Assume one of these operators is compact. Let* r *be a mapping from the natural numbers onto* $\{1, \dots, N\}$ *which assumes each value infinitely often, and let* $S_n = T_{r(n)} \cdots T_{r(1)}$. *Then* S_n *converges uniformly to the projection* Q *on the subspace* $\bigcap\{F(T_i) : 1 \le i \le N\}$.

Proof. We can assume $\|T_i\| = 1$ for each i, for otherwise S_n will converge uniformly to 0, as each map occurs infinitely often. Now by the remarks following Proposition 2, the projection Q commutes with each T_i. Moreover, we have $S_nQ = QS_n = Q$. Since the product of (W') operators is a (W') operator, the operators $(I - Q)T_i = T_i(I - Q)$, $1 \le i \le N$, are contractions on $(I - Q)X$ satisfying (W'), having (as we can assume) norm 1, and having no common nonzero fixed points. Since r assumes each value infinitely often, we can find a sequence $W_n = (I - Q)T_{r(k(n)+t(n))}T_{r(k(n)+t(n)-1)} \cdots T_{r(k(n))}$ of complete words in the $(I - Q)T_i$ whose index sets $\{k(n), \dots, k(n)+t(n)\}$ are mutually disjoint. Since one of the operators $(I - Q)T_i$ is compact, Lemma 4 provides us with a positive constant M such that $\|W_n\| \le M < 1$; for all n. Thus, if m is chosen so large that S_m contains k of the words W_n, then $\|(I - Q)S_m\| \le M^k$. Thus $\|S_m - Q\| = \|(I - Q)S_m\| \le M^k$, which goes to 0 as m (and therefore k) goes to ∞, proving the theorem.

This theorem improves upon the corresponding Hilbert space result in [7].

Remark. As mentioned earlier, condition (W) on each of N maps is not sufficient for strong convergence of random products. It can be shown that even the assumption of (S) on just two maps will not allow the conclusion of strong convergence of all random products, even when X is Hilbert. Hence an additional condition on the maps, such as compactness (as in Theorem 3), is needed. In this connection, see also [4].

We conclude this paper with an analog of Proposition 6.

Proposition 7. *Let* $\{T_j : 1 \le j \le N\}$ *be* N (W') *compact contractions on a smooth reflexive Banach space* X *with the KKP. Then the algebraic semigroup* $\mathbf{S} = \mathbf{S}(T_1, T_2, \dots, T_N)$ *generated by them satisfies condition* (S).

Proof. Suppose there exist bounded v_n such that $|v_n| - |W_nv_n| \to 0$ yet $(I - W_n)v_n$ does not converge strongly to zero. Then for a subsequence we may assume that $|(I - W_n)v_n| > \varepsilon > 0$. By Lemma 3 and Proposition 6 we know that

$$(17) \qquad\qquad (I - W_n)v_n \to 0 \text{ weakly}.$$

Since X is reflexive, we may assume

(18) $v_n \to u$ weakly

for a further subsequence. Hence (17) implies that

(19) $W_n v_n \to u$ weakly,

also.

Since there are only a finite number of maps, some fixed map T occurs infinitely often as a first letter of W_n. By dropping to a further subsequence, we may assume that T always follows v_n and that Tv_n converges strongly. In light of (18), $Tv_n \to Tu$ (strongly). Since $v_n - Tv_n \to 0$ strongly because T has (S), we see that $v_n \to Tu$ strongly. Now (18) implies that $Tu = u$. Hence $v_n \to u$ strongly. By dropping to an appropriate further subsequence, we may assume W_n always ends in the same (compact) map, and that $W_n v_n$ converges strongly. Necessarily $W_n v_n \to u$ strongly, because of (19). This contradicts our initial assumption that $|(I - W_n)v_n| > \varepsilon > 0$, completing the proof.

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